Calculus I

Challenge Homework Set I

April 8, 2025

Provide **handwritten** answers on a separate sheet of paper. Typed answers will not be accepted. For full credit correct answers should be clear, legible, include explanations for your reasoning, and show all relevant work. You are allowed to make use of outside resources, including the internet, and friends, but you must cite your sources.

i) In this problem we examine common precalculus pitfalls. Let x, y, z be real numbers. For each of the following expressions, determine whether the equality is true or false. In both cases you must explain your reasoning, and if the equality is false then change the right side of the expression so that the equality is true.

$a)(x+y)^2 = x^2 + y^2$

$b)\dfrac{x^2+y^2+z^2}{x^2+y^2} = 1 + \dfrac{z^2}{x^2+y^2}$

$c)xyz - (x^2 - z^3 + 3y) = xyz - x^2 - z^3 - 3y$

$d)\cos(x+y) = \cos x + \cos y$

$e)z^{x+y} = z^x + z^y$

$f)\log_z(xy) = \log_z x \cdot \log_z y$

We see that $a)$ this is false, the correct way to foil is:

$$(x+y)^2 = x^2 + 2xy + y^2$$

Next, $b)$ is true as:

$$\frac{x^2+y^2+z^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} + \frac{z^2}{x^2+y^2} = 1 + \frac{z^2}{x^2+y^2}$$

For $c)$, we see that this is false, as properly distributing the negative sign yields:

$$xyz - (x^2 - z^3 + 3y) = xyz - x^2 + z^3 - 3y$$

We see that $d)$ is false as the sum angle identity tells us that:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

Finally, $e)$ and $f)$ are false as:

$$z^{x+y} = z^x \cdot z^y \qquad \text{and} \qquad \log_z(x \cdot y) \log_z + \log_z y$$

ii) Using the squeeze theorem, and limit laws, calculate the following limits[1]:

a) $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$

b) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x}$

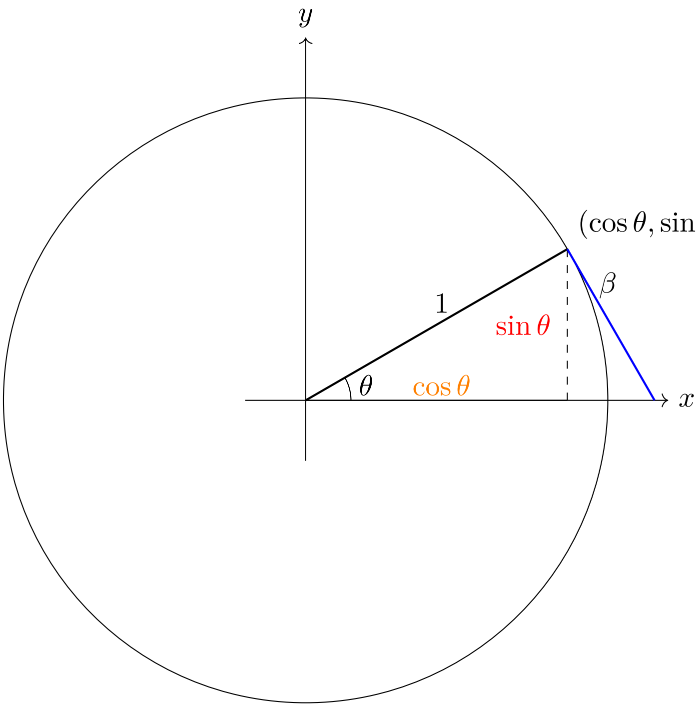
c) $\lim\limits_{x \to 0} x^2 \sin \dfrac{1}{x}$

d) $\lim\limits_{x \to 0} \dfrac{\sin 5x \cos x}{x}$

Now suppose that $f(x)$ is an arbitrary function, satisfying $\lim\limits_{x \to a} f(x) = L$. For n a positive whole number, and c a real number, show the following[2]:

e) $\lim\limits_{x \to a} (f(x))^n = L^n$ f) $\lim\limits_{x \to a} c \cdot f(x) = c \cdot L$

For a), we first examine the following diagram:



Note that the area of the triangle with side length $\sin\theta$ and $\cos\theta$ is given by $\frac{1}{2}\sin\theta\cos\theta$, which is less than the area of of the wedge cut out by the hypotenuse and the x-axis. Since this is a unit circle, the area of this wedge is $\frac{1}{2}\theta$. Finally this is less than the area of the big triangle made by the radius of the circle, the side labeled β, and the x-axis. We claim that the length of β is $\tan\theta$; indeed, the angle at the junction of the radius and β is a right angle, and so by definition $\tan\theta = \text{opp}/\text{adj}$ which is precisely $\beta/1$. The area of the largest triangle is thus $\frac{1}{2}\tan\theta$ so for $0 < \theta < \pi/2$:

$$\frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

[1]Hint: Your textbook covers two of these examples in great detail, and the other two are similar.
[2]Hint: Use the limit laws, the fact that the constant function $g(x) = c$ is continuous, and the fact that $h(x) = x^n$ is continuous.

Multiplying throughout by 2 we obtain that:

$$\sin\theta\cos\theta < \theta < \frac{\sin\theta}{\cos\theta}$$

Dividing by $\sin\theta$ through out:

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Taking reciprocals, we have that:

$$\frac{1}{\cos\theta} < \frac{\sin\theta}{\theta} < \cos\theta$$

Since $\lim_{x\to 0} 1/\cos x = 1$ and $\lim_{x\to 0}\cos\theta = 1$, which follows because both are continuous at $x = 0$, we have by the Squeeze theorem:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

For b), we observe that $\frac{1+\cos x}{1+\cos x} = 1$, hence:

$$\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}$$

By the limit law for multiplication, and part a), we have that:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x} =& \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x}\\ =&1\cdot 0\\ =&0\end{aligned}$$

For c), we observe that for all $x \neq 0$:

$$-1 < \sin\frac{1}{x} < 1$$

hence:

$$-x^2 < x^2\sin\frac{1}{x} < x^2$$

Since:

$$\lim_{x\to 0} \pm x^2 = 0$$

we have that by the Squeeze theorem:

$$\lim_{x\to 0} x^2\sin\frac{1}{x} = 0$$

For d), we know that:

$$\lim_{x\to 0}\cos x = 1$$

but can we figure out what:

$$\lim_{x\to 0}\frac{\sin 5x}{x}$$

is? The answer is yes, but we have to do something a little tricky. We have that:

$$\frac{\sin 5x}{x} = \frac{5 \sin 5x}{5x}$$

By part f) of this problem (which we will show momentarily), we have that:

$$\lim_{x \to 0} \frac{5 \sin 5x}{5x} = 5 \cdot \lim_{x \to 0} \frac{\sin 5x}{5x}$$

If we let $5x = u$, then this limit is the same as:

$$5 \cdot \lim_{u \to 0} \frac{\sin u}{u}$$

as x goes to zero, $5x$ goes to zero as well. It follows by a) that:

$$\lim_{x \to 0} \frac{\sin 5x}{x} = 5 \cdot \lim_{u \to 0} \frac{\sin u}{u} = 5$$

So by the multiplication law for limits:

$$\begin{aligned} \lim_{x \to 0} \frac{\sin 5x \cos x}{x} &= \lim_{x \to 0} \frac{\sin 5x}{x} \cdot \lim_{x \to 0} \cos x \\ &= 5 \cdot 1 = 5 \end{aligned}$$

For e), we set $g(x) = x^n$, and note that it is continuous for all real numbers. It follows by the rule for continuous functions:

$$\lim_{x \to a} (f(x))^n = \lim_{x \to a} g(f(x)) = g(\lim_{x \to a} f(x)) = g(L) = L^n$$

For f), we let $g(x) = c$, then b the limit rule for multiplication we have that:

$$\lim_{x \to a} g(x) \cdot f(x) = \lim_{x \to a} g(x) \cdot \lim_{x \to a} f(x) = c \cdot L$$

as desired.

iii) Let $f(x)$ and $g(x)$ be functions. We say that $f(x) < g(x)$ if:

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = \infty$$

Using this, order the following functions: $x!$, x, $\ln x$, x^e, e^x, $\sqrt{x}$, i.e. determine whether $\ln x < x$ or $x < \ln x$ for each function. Note that $x!$ is only defined for positive integers, and not all real numbers but we can still take limits like this if we restrict to the positive integers. What does this order tell you about the comparative growth rates of the functions?

We will use big number logic to demonstrate the the following order is correct:

$$\ln x < \sqrt{x} < x < x^e < e^x < x!$$

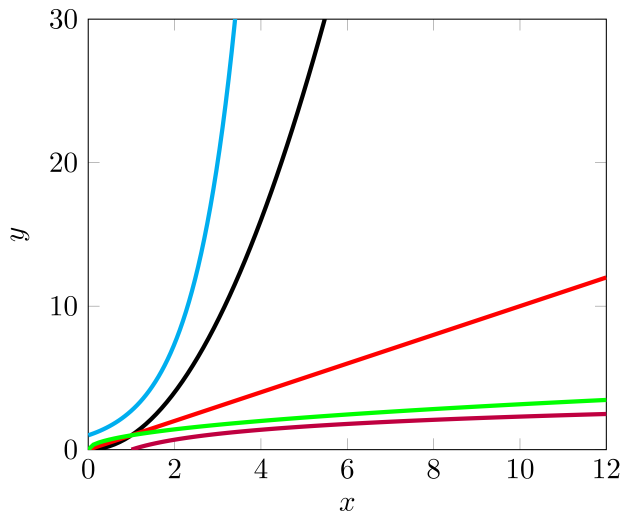
If we let x be a really big number, what is bigger?

$$1000! = 1000 \cdot 999 \cdot 998 \cdots 2 \cdot 1 \qquad \text{or} \qquad e^{1000} = \underbrace{e \cdot e \cdots e \cdot e}_{1000-\text{times}}$$

Clearly the left hand is much bigger, as almost every number in the product is greater than e. It follows that as x gets bigger, $x!$ gets much much bigger than e^x, hence:

$$\lim_{x\to\infty} \frac{x!}{e^x} = \infty$$

For the rest of the functions, we can make similar big number logic arguments, or just look at the following graph:



Where blue is e^x, black is x^2, red is x, green is $\sqrt{x}$, and purple is $\ln x$.

iv) In class we went over how one can use limits to calculate the speed of a ball falling off a building at a point t_0 in time. In this question, we consider a space ship flying away from earth; it's distance from earth for any $t > 0\,\text{s}$ is given in kilometers by the function:

$$x(t) = t^3$$

Calculate the speed of the spaceship when it is 8 km away from earth. Do not use the power rule, but instead mimic the procedure done on the first day of class.

The spaceship is 8 km away at $t_0 = 2\,\text{s}$. We have a $v_{\text{avg}}(h)$ function which gives us the average velocity of the spaceship over any interval of the form $[2, 2+h]$, it is given by:

$$v_{\text{avg}}(h) = \frac{(2+h)^3 - t^3}{h} = \frac{2^3 + 3\cdot 2h^2 + 3(2)^2h + h^3 - 8}{h}$$

For all h not equal to zero, this becomes:

$$v_{\text{avg}}(h) = 6h + 12 + h^2$$

So as h approaches 0, we have that velocity of the space ship is 12 km / s. In other words:

$$\lim_{h\to 0} v_{\text{avg}}(h) = 12$$

so the velocity is 12km/s

v) An object with an initial temperature of T_0 is placed in an environment of ambient temperature T_s. The temperature of the object as a function of time $t \geq 0$ is given by:

$$T(t) = T_s + (T_s - T_0)e^{-kt}$$

where k is some positive real number.

a) When does the temperature of the object reach one half of the temperature of the ambient environment?

b) Evaluate the limit $\lim_{x\to\infty} T(t)$.

c) Physically interpret your answer to *b*).

Note there was a typo in this problem, $T(t)$ should have given by:

$$T_s + (T_0 - T_s)e^{-kt}$$

but I took off no points one way or the other. Anyways, *b*) and *c*) do not depend on having the precisely correct form of $T(t)$. For *a*), we simply set:

$$T_s + (T_0 - T_s)e^{-kt} = \frac{1}{2}T_s$$

Note this only makes sense if $T_s > T_0$, as if $T_0 > T_s$ we obviously can't have the temperature of the object get colder than it's environment. Regardless, this implies that:

$$(T_0 - T_s)e^{-kt} = -\frac{1}{2}T_s$$

by subtracting T_s from both sides. We now divide by $T_0 - T_s$ to get:

$$e^{-kt} = -\frac{T_s}{2(T_0 - T_s)}$$

Taking a natural log of both sides, we have that:

$$-kt = \ln\left(-\frac{T_s}{2(T_0 - T_s)}\right)$$

hence:

$$t = -\frac{\ln\left(-\frac{T_s}{2(T_0-T_s)}\right)}{k}$$

For *b*), we see that using big number logic, as t gets really big, T_s is left unchanged, but $(T_0 - T_s)e^{-kt}$ gets closer and closer to zero because e^{-kt} gets closer and closer to zero, because $e^{-kt} = 1/e^{kt}$. It follows that:

$$\lim_{x\to\infty} T(t) = T_s + (T_0 - T_s)\cdot 0 = T_s$$

Finally, for *c*), we interpret this answer as follows: eventually, as long as enough time passes, the temperature of the object will reach the temperature of it's surroundings.